Coordinate space wave function from the Algebraic Bethe Ansatz for the inhomogeneous six-vertex model.

A.A. Ovchinnikov

Institute for Nuclear Research, RAS, Moscow, 117312, Russia

Abstract

We derive the coordinate space wave function for the inhomogeneous six-vertex model from the Algebraic Bethe Ansatz. The result is in agreement with the result first obtained long time ago by Yang and Gaudin in the context of the problem of one-dimensional fermions with δ- function interaction.

1. Introduction.

The solution of the six-vertex model with an arbitrary inhomogeneity parameters was an important step in the theory of integrable systems. In the rational case the solution was given independently by Yang [1] and Gaudin [2] using the generalization of the coordinate Bethe ansatz (for a review see for example [3]). Later the solution was simplified drastically in the framework of the Algebraic Bethe Ansatz method (for example see [4]). However, since up to now the form of the coordinate space wave function of the eigenstates of the transfer matrix in the framework of the Algebraic Bethe Ansatz was not investigated, the connection of this two approaches remains obscure. The goal of the present letter is to fill this gap and derive the coordinate space wave function of the Bethe eigenstate given by the Algebraic Bethe Ansatz. We find the complete agreement with the results of ref's [1], [2].

The solution of this problem is achieved with the help of the so called Factorizing operator. This operator introduced in ref.[5] plays an important role in the solution of Quantum Inverse Scattering problem [6], calculations of the correlation functions in spin chains [7] and scalar products [8]. The calculation of the wave function which looks hopeless in terms of the usual Algebraic Bethe Ansatz operators, is quite simple in the F-basis for the operators in the auxiliary space given by the tensor product of M (the number of up-spins) spin 1/2 auxiliary spaces. In Section 2 we introduce the notations and briefly discuss the definition and the properties of the Factorizing operator. We calculate the wave function of Bethe eigenstate in Section 3. Finally in Section 3 we present the conclusion.

2. Algebraic Bethe Ansatz and the F-basis.

We consider in this letter the transfer matrices corresponding to both rational and trigonometric regimes of the six-vertex model. In the present section we diagonalize the operator A

and introduce the factorizing operator. Let us fix the notations: the normalization of basic S - matrix, the definition of monodromy matrix and write down the Bethe Ansatz equations. For the rational case the S- matrix has the form $S_{12}(t_1,t_2) = t_1 - t_2 + \eta P_{12}$, where P_{12} is the permutation operator. In general trigonometric case it can be written as

$$S_{12}(t_1,t_2) = \left(\begin{array}{cccc} a(t) & 0 & 0 & 0 \\ 0 & c(t) & b(t) & 0 \\ 0 & b(t) & c(t) & 0 \\ 0 & 0 & 0 & a(t) \end{array} \right)_{(12)} , \qquad t = t_1 - t_2.$$

One can choose the normalization $a(t) = 1$ so that the functions $b(t)$ and $c(t)$ become

$$\tilde{c}(t) = \frac{\phi(t)}{\phi(t+\eta)}, \quad \tilde{b}(t) = \frac{\phi(\eta)}{\phi(t+\eta)},$$

where $\phi(t) = t$ for the rational case and $\phi(t) = \sin(t)$ for the trigonometric case. With this normalization the S-matrix satisfies the unitarity condition $S_{12}(t_1,t_2)S_{21}(t_2,t_1) = 1$. The monodromy matrix is defined as

$$T_0(t,\{\xi\}) = S_{10}(\xi_1,t)S_{20}(\xi_2,t)...S_{L0}(\xi_N,t),$$

where ξ_i are the inhomogeneity parameters and L is the length of the lattice. We define the operator entries in the auxiliary space (0) as follows:

$$\langle\beta|T_0|\alpha\rangle = \left(\begin{array}{cc} A(t) & B(t) \\ C(t) & D(t) \end{array} \right)_{\alpha\beta}; \qquad \alpha,\beta = (1,2) = (\uparrow;\downarrow).$$

We denote throughout the paper $(\uparrow;\downarrow) = (1;0)$ so that the pseudovacuum (quantum reference state) $|0\rangle = |\{00...0\}_L\rangle$. The triangle relation (Yang-Baxter equation) reads:

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}, \quad R_{00'}T_0T_{0'} = T_{0'}T_0R_{00'}; \quad R_{00'} = S_{0'0}.$$

The action of the operators on the pseudovacuum is: $A(t)|0\rangle = a(t)|0\rangle$ $(a(t) = \prod_\alpha \tilde{c}(\xi_\alpha - t))$, $D(t)|0\rangle = |0\rangle$, $C(t)|0\rangle = 0$. The Bethe Ansatz equations for the eigenstate of the Hamiltonian $\prod_{i=1}^M B(q_i)|0\rangle$ and the corresponding eigenvalue of the transfer - matrix $Z(t) = A(t) + D(t)$ are

$$a(q_i) = \prod_{\alpha\neq i} \tilde{c}(q_\alpha - q_i)(\tilde{c}(q_i - q_\alpha))^{-1}, \qquad \Lambda(t,\{q_\alpha\}) = a(t)\prod_{\alpha=1}^M \tilde{c}^{-1}(q_\alpha - t) + \prod_{\alpha=1}^M \tilde{c}^{-1}(t - q_\alpha),$$

where q_α are the solution of Bethe Ansatz equations.

In the present paper we will use the monodromy matrix in the F -basis, the basis obtained with the help of the factorizing operator F introduced in ref.[5]. One can construct the

operator $F = F_{1\ldots L}$ which diagonalizes the operator $A(t)$ [5],[6],[8] ($A^F(t) = F^{-1}A(t)F$). The diagonal operator $A^F(t)$ has the following form:

$$A^F(t) = \prod_{i=1}^{L} \left(\tilde{c}(\xi_i - t)(1 - n_i) + n_i\right). \tag{1}$$

where we denote by n_i the operator of the number of particles (hard-core bosons, corresponding to the up-spin) at the site i. Let us briefly mention some of the properties of the operator F. The explicit form of the operator F is

$$F_{12\ldots N} = \hat{F}_1\hat{F}_2\ldots\hat{F}_L, \qquad \hat{F}_i = (1 - \hat{n}_i) + T_i\hat{n}_i, \tag{2}$$

where $\hat{n}_i$ is the operator of the number of particles (spin up) at the site i and the operator T_n is given by the equation

$$T_n = S_{n+1,n}S_{n+2,n}\ldots S_{Ln}.$$

One can obtain the following formulas for the matrix elements of the operator F [8] in the following form:

$$F_{\{m\}\{n\}} = \langle\{m\}|B(\xi_{n_1})B(\xi_{n_2})\ldots B(\xi_{n_M})|0\rangle,$$

where the sets of coordinates $\{m\}$ and $\{n\}$ label the positions of the occupied sites. The similar expression can be obtained for the inverse operator F^{-1}. Apart from diagonalizing the operator $A(t)$, the operator F is the factorizing operator [5] in the following sense. For any permutation of indices $\sigma \in S_L$ (S_L - is the group of permutations) we have the equation $F = F^\sigma R^\sigma$, where $F^\sigma_{12..L} = F_{\sigma1\sigma2..\sigma L}$ (including the permutation of the inhomogeneity parameters ξ_i) and $R^\sigma_{1\ldots L}$ is the operator constructed from the S- matrices defined in such a way that for the permutation of the monodromy matrix $T^\sigma_0 = T_{0,\sigma1\sigma2..\sigma L}$ we have $T^\sigma_0 = (R^\sigma)^{-1}T_0R^\sigma$. For the particular permutation $\sigma(\{n\})$ such that $\sigma1 = n_1, \ldots \sigma M = n_M$ ($n_1 < n_2 < \ldots < n_M$) the factorization condition is represented as $F(F^{\sigma(\{n\})})^{-1} = T_{n_1}..T_{n_M}$. To prove the factorizing property of the operator (2) it is sufficient to consider only one particular permutation, for example, the permutation $(i, i+1)$, since all the other can be obtained as a superposition of these ones for different i. One can show, that $F = S_{i+1,i}F^{(i,i+1)}$, which evidently proves the factorization property.

The matrix elements of the operators $B(t)$ and $C(t)$ in the F - basis: $B^F(t) = F^{-1}B(t)F$ (and the same for $C(t)$) have the following form

$$B^F(t) = \sum_i \sigma_i^\dagger\ \tilde{b}(\xi_i - t)\prod_{k\neq i}\Big(\tilde{c}(\xi_k - t)(\tilde{c}(\xi_k - \xi_i))^{-1}(1 - n_k) + n_k\Big). \tag{3}$$

$$C^F(t) = \sum_i \sigma_i^-\ \tilde{b}(\xi_i - t)\prod_{k\neq i}\Big(\tilde{c}(\xi_k - t)(1 - n_k) + (\tilde{c}(\xi_i - \xi_k))^{-1}n_k\Big). \tag{4}$$

The operators (3) and (4) are quasilocal i.e. they describe flipping of the spin on a single site with the amplitude depending on the positions of the up-spins on the other sites of the chain.

The operator $D^F(t)$ can be found, for example, using the quantum determinant relation and has a (quasi)bilocal form.

For our calculations performed in the next Section it will be useful to define the operators $B_i(t)$, $i = 1 \dots L$ creating the particle (spin up) at the site i defined as $B^F(t) = \sum_i B_i(t)$. According to Eq.(3) we have explicitly:

$$B_i(t) = \sigma_i^\dagger \, \tilde{b}(\xi_i - t) \prod_{k \neq i} \Big(\tilde{c}(\xi_k - t)(\tilde{c}(\xi_k - \xi_i))^{-1}(1 - n_k) + n_k \Big) . \tag{5}$$

It is important that the operators $B_i(t)$ have a very simple commutational relations with the operator $A^F(t')$ (in contrast to the well known commutational relations of $B(t)$ and $A(t')$). In fact we have:

$$B_i(t) A^F(t') = \tilde{c}(\xi_i - t') A^F(t') B_i(t). \tag{6}$$

Equations (6) will be used later in the next Section.

Let us mention here that the operators $B_i = B_i(0)$ have the following commutational relations:

$$B_i B_j = \tilde{S_{ij}} B_j B_i, \qquad \tilde{S_{ij}} = \frac{\tilde{c}(\xi_i)\tilde{c}(\xi_j - \xi_i)}{\tilde{c}(\xi_j)\tilde{c}(\xi_i - \xi_j)}. \tag{7}$$

From the equation (7) it easy to see that the matrices $\tilde{B}_i$ ($B_i = \tilde{B}_i A^F$) and A^F taken in the auxiliary space corresponding to the lattice of M sites $0_1, \dots 0_M$ with the corresponding spectral parameters give the explicit realization of the matrices entering the so called Matrix Product Ansatz for the XXZ spin chain [9].

3. Coordinate space wave function.

We have to calculate the coordinate-space wave function for the inhomogeneous six-vertex model which defines the eigenstate $|\phi\rangle$ of the transfer-matrix according to the equation

$$|\phi\rangle = \sum_{x_1, \dots x_M} \psi(x_1, \dots x_M) |x_1, \dots x_M\rangle,$$

where the sum is over the configurations of particles (up-spins) $x_i \neq x_j$ for $i \neq j$, and $|x_1, \dots x_M\rangle$ is the state with the occupied sites $1 \leq x_i \leq L$. The wave function is given by the following scalar product:

$$\psi(x_1, \dots x_M) = \langle x_1, \dots x_M | B(q_1) B(q_2) \dots B(q_M) | 0 \rangle, \tag{8}$$

where q_i are the parameters which obey the Bethe ansatz equations. We consider the wave function (8) in the sector $x_1 < x_2 < \dots < x_M$.

Using the definition of the operators $B(q_i)$ in terms of the S- matrices, reordering the S-matrices in the product $\prod_i B(q_i)$, we rewrite the equation (8) in terms of the new monodromy matrices $T_i = S_{i0_1} S_{i0_2} \dots S_{i0_M}$, $i = 1, \dots L$, depending on the site i and acting in the new quantum space $(0_1, \dots 0_M)$, as follows:

$$\psi(x_1, \dots x_M) = \langle 0 | \langle x_1 \dots x_M | T_1 T_2 \dots T_L | 0 \rangle_L | \{11...1\}_M \rangle,$$

where the second average corresponds to the space $(0_1, \ldots 0_M)$. We evaluate the average in the quantum space $1, 2, \ldots L$, use the symmetry $0 \leftrightarrow 1, A \leftrightarrow D, B \leftrightarrow C$ and transform the operators acting in the space $(0_1, \ldots 0_M)$ to the operators in the F-basis. One should also take into account that the action of the operator F to the vacuum is trivial: $F|0\rangle = |0\rangle$.

Thus we have to calculate the following average over the states on the lattice with M sites $0_1, \ldots 0_M$:

$$\psi(x_1, \ldots x_M) = \langle \{11...1\}_M | A^F(\xi_1) A^F(\xi_2) ... B^F(\xi_{x_1}) ... B^F(\xi_{x_M}) ... A^F(\xi_L)|0\rangle, \tag{9}$$

where the operators $B^F(\xi_{x_i})$, $i = 1, \ldots M$ are located at the positions $x_1, \ldots x_M$ and the operators A^F and B^F correspond to the new transfer matrix of the form $T_i = S_{i0_1} S_{i0_2} \ldots S_{i0_M}$. The factorizing operator and the operators in the F-basis are also defined in the space $(0_1, \ldots 0_M)$. In contrast with the similar expression with the operator A and B in the usual basis, one can obtain the compact expression for the average (9) with the operators in the F-basis.

Since the matrix A^F is diagonal, the next step to obtain the wave function is to use the equation (5) to rewrite the average (9) as a sum over the permutations. In fact one obtains:

$$\psi(x_1, \ldots x_M) = \sum_{P \in S_M} \langle \{11...1\}_M | A^F(\xi_1) A^F(\xi_2) ... B_{P1}(\xi_{x_1}) ... B_{PM}(\xi_{x_M}) ... A^F(\xi_L)|0\rangle. \tag{10}$$

Now using eq.(6) we commute all the operators $B_{Pi}(\xi_{x_i})$ to the left. The action of the operators $A^F(\xi_l)$, $l \neq x_i$ to the right- hand state (to the vacuum) is known, so we obtain the following compact expression (in the sector $x_1 < x_2 < \ldots < x_M$):

$$\sum_{P \in S_M} \prod_{l_1 < x_1} \frac{1}{\tilde{c}(\xi_{l_1} - q_{P1})} \prod_{l_2 < x_2, l_2 \neq x_1} \frac{1}{\tilde{c}(\xi_{l_2} - q_{P2})} \cdots \prod_{l_M < x_M, l_M \neq x_1, \ldots x_{M-1}} \frac{1}{\tilde{c}(\xi_{l_M} - q_{PM})}$$

$$\prod_{l=1, l \neq x_i}^{M} \prod_{j=1}^{M} \tilde{c}(\xi_l - q_j) \langle \{11...1\}_M | B_{P1}(\xi_{x_1}) ... B_{PM}(\xi_{x_M})|0\rangle. \tag{11}$$

Note that one have to change the signs of all the spectral parameters in the relation (6) and the formula for the action of the operators $A^F(\xi_i)$ to the vacuum due to the definition of the new monodromy matrices T_i acting in the new quantum space $(0_1, \ldots 0_M)$. The last step is to evaluate the average at the end of the last equation. Acting consequently by the operators B_{Pi} to the right we obtain the expression:

$$\Phi_M(\xi_x, q|P) = \langle \{11...1\}_M | B_{P1}(\xi_{x_1}) ... B_{PM}(\xi_{x_M})|0\rangle$$

$$= \prod_{i=1}^{M} \tilde{b}(\xi_{x_i} - q_{Pi}) \prod_{i>j} \tilde{c}(\xi_{x_i} - q_{Pj}) \prod_{i>j} \frac{1}{\tilde{c}(q_{pi} - q_{Pj})}. \tag{12}$$

The sum over the permutations P of this expression gives the partition function of the six-vertex model with domain-wall boundary conditions $\sum_P \Phi_M(\xi_x, q|P) = \Phi_M(\xi_x, q)$. It is interesting to obtain the determinant representation for this function $\Phi_M(\xi_x, q)$ [10],[11] starting from the representation (12). In fact, representing the sum over the permutations in

$\sum_P \Phi_M(\xi_x, q|P)$ as $\sum_{i=1}^M \sum_{P:PM=i} \Phi_M(\xi_x, q|P)$, and considering separately the dependence on the variables ξ_{x_M} and $q_{PM} = q_i$, we obtain the following recurrence relation for the function $\Phi_M(\xi_x, q)$:

$$\Phi_M(\{\xi_x\}, \{q\}) = \sum_{i=1}^M \tilde{b}(\xi_{x_M} - q_i) \prod_{\alpha \neq i} \frac{\tilde{c}(\xi_{x_M} - q_\alpha)}{\tilde{c}(q_i - q_\alpha)} \Phi_{M-1}(\{\xi_{x_\alpha}\}_{\alpha \neq M}, \{q_\beta\}_{\beta \neq i}),$$

which coincides with the well known recurrence relation which determines the determinant expression for $\Phi_M(\xi_x, q)$ (for example, see Appendix B of ref.[8]).

Substituting the expression (12) into the equation (11) and performing the cancellations of similar terms we easily obtain the following result:

$$\psi(x_1, \ldots x_M) = \sum_P A(P) \phi_{P1}(x_1) \phi_{P2}(x_2) \ldots \phi_{PM}(x_M), \tag{13}$$

where the functions $\phi_{Pi}(x_i)$ and the amplitude $A(P)$ are equal to

$$\phi_{Pi}(x_i) = \prod_{l > x_i}^L \tilde{c}(\xi_l - q_{Pi}) \tilde{b}(\xi_{x_i} - q_{Pi}), \qquad A(P) = \prod_{i>j}^M \frac{1}{\tilde{c}(q_{Pi} - q_{Pj})}. \tag{14}$$

The expression (13) for the wave function is the final result of the present paper. In order to compare this result with the results obtained long time ago by the other authors it is useful to represent the function $\phi_{Pi}(x_i)$ in the following form:

$$\phi_{Pi}(x_i) = \left(\prod_{l=1}^L \tilde{c}(\xi_l - q_{Pi}) \right) \left(\tilde{c}^{-1}(\xi_{x_i} - q_{Pi}) \tilde{b}(\xi_{x_i} - q_{Pi}) \right) \prod_{l < x_i} \tilde{c}^{-1}(\xi_l - q_{Pi}). \tag{15}$$

Thus, up to the normalization factor the wave function (13) coincides with the wave function obtained by the other authors (for example, see [3], [1]).

Let us obtain the Bethe ansatz equations for the inhomogeneous six-vertex model starting from the wave function (13). We have to continue the wave function out of the interval $(1, L)$ and impose the periodic boundary conditions of the form:

$$\psi(x_1, \ldots x_M) = \psi(x_1 + L, x_2, \ldots x_M) = \psi(x_2, \ldots x_M, x_1 + L) \tag{16}$$

in the sector $x_1 < x_2 < \ldots < x_M$. Substituting the wave function (13) into the equation (16) we obtain:

$$\sum_P A(P) \phi_{P1}(x_1) \ldots \phi_{PM}(x_M) = \sum_P A(PC) \phi_{P1}(x_1 + L) \phi_{P2}(x_2) \ldots \phi_{PM}(x_M),$$

where C - is the cyclic permutation ($C1 = 2$, $C2 = 3 \ldots . CM = 1$). This equation gives the following condition for the amplitude $A(P)$ (14):

$$A(P)/A(PC) = \prod_{l=1}^L \tilde{c}^{-1}(\xi_l - q_{P1}). \tag{17}$$

One can easily verify that the equations (17) are equivalent to the usual Bethe ansatz equations for the six-vertex model with the inhomogeneity parameters ξ_l.

In conclusion, we derived the coordinate space wave function for the inhomogeneous six-vertex model from the Algebraic Bethe Ansatz. Our result is in agreement with the result first obtained by Yang and Gaudin for the eigenstate of the transfer matrix of the six-vertex model in the rational case.

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